The Covering Codes Problem

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GAANN Computational Techniques
Three players enter a room and a red or blue hat is placed on each person’s head. The color of each hat is determined by a coin toss, with the outcome of one coin toss having no effect on the others. Each person can see the other players’ hats but not his own.

No communication of any sort is allowed, except for an initial strategy session before the game begins. Once they have had a chance to look at the other hats, the players must simultaneously guess the color of their own hats or pass. The group shares a hypothetical $3 million prize if at least one player guesses correctly and no players guess incorrectly.

The same game can be played with any number of players. The general problem is to find a strategy for the group that maximizes its chances of winning the prize.

Sara Robinson
New York Times, April 10, 2001
## CONFIGURATIONS

<table>
<thead>
<tr>
<th>PLAYER 1</th>
<th>PLAYER 2</th>
<th>PLAYER 3</th>
<th>OUTCOME</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>LOSE</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>WIN</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>WIN</td>
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<td>0</td>
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<td>1</td>
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<td>0</td>
<td>WIN</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>LOSE</td>
</tr>
</tbody>
</table>

The two colors of hats are represented by 0, 1.
Another interpretation of the solution...

- Configuration space is covered by ‘spheres’ of radius 1
- Strategy ensures that teams lose at centers of spheres
- Is 75% the best possible outcome?
- Is this the only 75% strategy?
- Multiple solutions are geometrically apparent
- Could generalize rules of game to any number of players
- Strategy remains the same- avoid certain ‘centers’ in configuration space
- Picture generalizes to hypercube
For the $n$ player game, finding the optimal solution is equivalent to the following problem:

**COVERING CODES PROBLEM**

What is the smallest number of spheres of radius 1 required to cover the space of all $n$ digit binary numbers

- This number is denoted $K(n,1)$
- We have seen that $K(3,1)=2$  
- In general, the optimal winning percentage is $1 - K(n,1)/2^n$
A covering is a collection of ‘centers’ such that all other points are within distance 1.

In binary case called a covering code.

Intuition: boundary makes it difficult to calculate (or even estimate).

In general can expect intersections, even in optimal solutions.

Perfect coverings only possible when $n = 2^k - 1$ (Hamming Codes).
**Known Results: bounds on K(n,1)**

<table>
<thead>
<tr>
<th>Number of Digits, n</th>
<th>Minimum Code Size, K(n,1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
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<tr>
<td>3</td>
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<td>8</td>
<td>32</td>
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<td>9</td>
<td>57-62</td>
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<tr>
<td>10</td>
<td>105-120</td>
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<tr>
<td>11</td>
<td>180-192</td>
</tr>
<tr>
<td>12</td>
<td>342-380</td>
</tr>
<tr>
<td>13</td>
<td>598-736</td>
</tr>
<tr>
<td>14</td>
<td>1172-1408</td>
</tr>
<tr>
<td>15</td>
<td>2048</td>
</tr>
<tr>
<td>16</td>
<td>4096</td>
</tr>
</tbody>
</table>

**Why is it so difficult?**

- Lower bounds by proofs
- Upper bounds by computer search

**In the 127 player hat game, chances of winning are 127/128 !!!**

Source: Simon Litsyn
Covering Codes Applet:

home.attbi.com/~paulandlynn/paul/math.html
1953 Metropolis - proved samples system with a Quasi-Boltzman distribution

1983 Kirkpatrick - first proposed as general optimization algorithm

FOR COVERING CODES

A ‘good’ initial covering is chosen by theoretical construction

Simulated annealing used in tricky way to improve the covering lower bound

Randomly perturb the proposed solution
- If value is lower ACCEPT
- If value is higher ACCEPT with probability $\exp(-df/T)$
ERROR DETECTION

Easy to check if a single error was made: 010110 has checksum 1

To correct result requires retransmission

Not acceptable in certain situations: communication with spacecraft

AN ERROR CORRECTING CODE . . . .

Agree beforehand:

\[ \begin{align*}
000 & \leftrightarrow A \\
111 & \leftrightarrow B
\end{align*} \]

\[ \text{ABAAB} \leftrightarrow 000111000000111 \]

If 010 received assume intended meaning is A

Data transmission efficiency is 33% for this code