Stochastic unilateral free vibration of an in-plane cable network

Gian Felice Giaccu a, Bernardo Barbiellini b, Luca Caracoglia c,∗

a Department of Architecture, Design and Urban Planning, University of Sassari, 07041 Alghero, Italy
b Department of Physics, Northeastern University, Boston, MA 02115, USA
c Department of Civil and Environmental Engineering, 400 Snell Engineering Center, Northeastern University, 360 Huntington Avenue, Northeastern University, Boston, MA 02115, USA

A R T I C L E   I N F O

Article history:
Received 20 July 2014
Received in revised form 25 October 2014
Accepted 1 December 2014
Handling Editor: W. Lacarbonara
Available online 27 December 2014

A B S T R A C T

Cross-ties are often used on cable-stayed bridges for mitigating wind-induced stay vibration since they can be easily installed on existing systems. The system obtained by connecting two (or more) stays with a transverse restrainer is designated as an “in-plane cable-network”. Failures in the restrainers of an existing network have been observed. In a previous study [1] a model was proposed to explain the failures in the cross-ties as being related to a loss in the initial pre-tensioning force imparted to the connector. This effect leads to the “unilateral” free vibration of the network. Deterministic free vibrations of a three-cable network were investigated by using the “equivalent linearization method”.

Since the value of the initial vibration amplitude is often not well known due to the complex aeroelastic vibration regimes, which can be experienced by the stays, the stochastic nature of the problem must be considered. This issue is investigated in the present paper. Free-vibration dynamics of the cable network, driven by an initial stochastic disturbance associated with uncertain vibration amplitudes, is examined. The corresponding random eigen-value problem for the vibration frequencies is solved through an implementation of Stochastic Approximation, (SA) based on the Robbins–Monro Theorem. Monte-Carlo methods are also used for validating the SA results.

© 2014 Elsevier Ltd. All rights reserved.

1. Introduction

1.1. Context and motivation

Inclined stays on modern cable-stayed bridges are susceptible to wind-induced oscillations (e.g., [2,3]) due to their lengths and slenderness. Several models exist for predicting the loading mechanisms. Typical examples are “dry galloping” (e.g., [4,5]) causing large-amplitude oscillation and “rain-wind induced” oscillation (see, for example, [6] for a recent review). Moreover, other excitation mechanisms can influence the dynamics of the stays, such as various cable-deck interaction phenomena, linear or nonlinear (e.g., [7–13]). Nevertheless, the loading estimation on stay cables still remains an open and partially unresolved issue.

∗Corresponding author. Tel.: +1 617 373 5186; fax: +1 617 373 4419.
E-mail addresses: gf.giaccu@uniss.it (G.F. Giaccu), bba@neu.edu (B. Barbiellini), lucac@coe.neu.edu (L. Caracoglia).

http://dx.doi.org/10.1016/j.jsv.2014.12.004
0022-460X/© 2014 Elsevier Ltd. All rights reserved.
The researchers. A new model was developed by the authors [27] for the prediction of oscillations on a cable network in suppression of stay vibration in the plane orthogonal to the plane of the stays is limited (e.g., [24, 25]) dynamic models, based on linear taut-cable theory [23], was proposed to study the in-plane free vibration of cable networks and for nonlinear dampers (e.g., [17]).

The model was employed [1] to find the minimum level of performance coefficient of the cross-tie segment, installed between stays \( j \) and \( j+1 \) with respect to \( \alpha_\epsilon \), evaluated at \( \alpha_\epsilon \equiv \pi_\epsilon \) first derivative of the function \( \Delta K_j \) (j = 1) with respect to \( \alpha_\epsilon \), evaluated at \( \alpha_\epsilon = \pi_\epsilon \) second derivative of the function \( \Delta K_j \) (j = 1) with respect to \( \alpha_\epsilon \), evaluated at \( \alpha_\epsilon = \pi_\epsilon \) exponent in the formula of the gain parameter \( a_\epsilon \) absolute (Eq. (A.3)) and relative (Eq. (A.4)) approximation error due to the hypothesis \( E[\Delta K_j(a_\epsilon)] \cong \Delta K_j(E[a_\epsilon])] \) dimensionless vibration amplitude \( \lambda \) dimensionless amplitude, element \( q \) of the random sequence upper limit value of the \( \lambda \) random variable noisy function of the variable \( \alpha_\epsilon \) evaluated by the SA algorithm at step \( q \)

Subscripts

\( j \) generic stay-cable index

\( q \) index designating the element of a random sequence, also used in recursive SA formulas to designate the iteration step

\( m \) sample size of the random sequence of \( \lambda \)

Cross-ties are often employed for vibration reduction in the stays since their installation on existing systems is simple. Therefore, accurate prediction of cable network dynamics is of great relevance for the design of mitigation systems on cable-stayed bridges. Even though the emphasis of this study is on cable-cross-tie systems, we provide a short description of other methods for vibration control for the sake of completeness. Mitigation may also be achieved by using, as an alternative, dynamic connectors, i.e., restrainers with nonlinear restoring effects [26]. More recently, nonlinear dynamic response of stayed bridges. Even though the emphasis of this study is on cable-cross-tie systems, we provide a short description of other methods for vibration control for the sake of completeness. Mitigation may also be achieved by using, as an alternative, dynamic connectors, i.e., restrainers with nonlinear restoring effects [26].

Finite-element analysis has traditionally been used to study the dynamics of cross-ties [21, 22]. An analytical model, based on linear taut-cable theory [23], was proposed to study the in-plane free vibration of cable networks and for predicting the response of real systems (e.g., [24]). This linear analytical formulation has also been considered and used by other investigators to further examine the cross-tie adequacy in controlling wind-related oscillation (e.g., [25]). Since suppression of stay vibration in the plane orthogonal to the plane of the stays is limited (e.g., [24, 25]) dynamic models, which include out-of-plane behavior, are not necessary.

To date few studies have addressed the relevant question of cross-tie performance in the presence of nonlinear dynamic connectors, i.e., restrainers with nonlinear restoring effects [26]. More recently, nonlinear dynamic response of a network at incipient failure, due to snapping or slackening of the transverse restrainers, has gained the attention of the researchers. A new model was developed by the authors [27] for the prediction of oscillations on a cable network in the presence of a nonlinear restoring-force effect in the connectors. The “Equivalent Linearization Method” (ELM) was used [27, 28]. It was found that, by comparing the ELM solution to direct time-domain integration, the ELM is still accurate for predicting the free vibration. In a previous study [1] the nonlinear effect associated with the incipient slackening in the restrainer due to the loss of the pre-tensioning force, initially imparted to the restrainers, was analyzed in more detail. “Unilateral behavior” was employed to simulate extreme conditions in the restrainer at slackening. This model can reproduce the unilateral restoring-force trend in the cross-ties by using a dimensionless pre-tensioning parameter \( r_{0.1} > 0 \), which sets the initial level of pre-stressing force in the connector. The quantity \( r_{0.1} \) was coined to evaluate the unilateral performance of a spring-type mechanistic model of the cross-tie (e.g., [24]). The model operates by linearization of the system of differential dynamic equations (ELM). The algorithm estimates a “performance coefficient” of the cross-tie (\( \Delta K_j \)) as a function of vibration amplitude \( \lambda \) of the cable network and of \( r_{0.1} \). The model was employed [1] to find the minimum level of \( r_{0.1} \), needed to preserve linearity in the cross-tie response, depending on the vibration amplitude parameter \( \lambda \) of the system. It must be noted that the amplitude parameter \( \lambda \) was coined in [1] to indicate a dimensionless ratio between maximum modal amplitudes in the stays during unilateral free vibration, induced by nonlinear behavior in the cross-tie. This quantity must not be confused with...
the Irvine parameter [23], used to describe the influence of sag on the vibration frequencies of non-shallow cables; as taut-
cable theory was utilized to simulate stay vibration, sag effects in the stays was neglected.

As shown in a previous study [1], the true free-vibration response exhibits periodic transitions from the reference
scenario with cross-tie responding linearly to a “new” system after slackening, in which the stays are partially disconnected
and vibrate according to their native frequencies. After cross-tie slackening, the ELM operates by evaluating a “fictitious
reduction” of cross-tie stiffness, which allows detecting the malfunctioning in the network as a function of the generalized
vibration amplitude and the initial pre-tensioning force in the cross-tie. It was shown [1] that, by comparing a series of
numerical simulations using a lumped-mass model of a three-stay cable network, the in-plane dynamics can become
nonlinear. Nevertheless, it was observed that the ELM can still be used to analyze the dynamics of the system by introducing
the concept of performance coefficient as a damage indicator. The analysis of the lumped-mass model simulations also
enabled to identify the linear regime of the network. In particular, the vibration amplitudes needed to ensure the adequate
behavior of the system were verified with the performance coefficient.

1.2. Relevance of the stochastic analysis and objectives of the study

In this paper, the ELM [1] is combined with a stochastic approach able to illustrate the effect of uncertainty in the
amplitude parameter $\lambda$ on the cable network behavior. The study aims at examining the problem of inadequate knowledge
of the vibration mechanism, influenced by irregular wind load features. This lack of knowledge can be included by stochastic
perturbation, operating on the vibration amplitude parameter $\lambda$, which controls the deterministic solution [1]. The
randomization of $\lambda$ is used to simulate the lack of understanding of the vibration dynamics; this hypothesis indirectly
reflects uncertainty in the frequency estimation, which is a consequence of the inability to describe the various wind-related
aeroelastic mechanisms. The stochastic feature is simulated by parametric perturbation to the linear dynamic problem,
assuming that no or little vibration is observed in the absence of fluid-induced oscillation. A uniform probability distribution
is employed to describe the variability of $\lambda$. This assumption reproduces the epistemic uncertainty by preserving simplicity
in the modeling. The uniform probability distribution best represents the current state of knowledge of the various vibration
regimes. The parameter $\lambda$ must not be interpreted as an external random “noise”; therefore the use of other probability
distributions for describing the random $\lambda$, such as the Gaussian distribution, might be not a good choice.

After the selection of a stochastic amplitude $\lambda$, the ELM problem becomes an equivalent random eigen-value problem, in
which the frequencies of the cable network are affected by a noisy $\lambda$ and must be determined in a stochastic setting. The
random eigen-value solution is extracted through an implementation of the Stochastic Approximation (SA) [29], which is
based on the Robbins–Monro Theorem [29,30]. The SA approximates the solution of an equation, observed through noise.
This method is “adaptive” and designed for uncertain or stochastic environments, where it allows for tracking the “average”
or the “typical” features in such an environment. It is incremental, i.e., it makes small changes in each step, which ensures a
smooth convergence to the solution [31,32]. Other techniques have been used to solve random eigenvalue problems.
Representative examples, among many, include crossing theory of random processes [33], stochastic reduced-order models
describing the characteristic polynomial associated with the eigen-values [34], Polynomial Chaos expansion [35] and
collocation methods [36]. Nevertheless, the SA is simpler and can be easily adapted to various cases involving complex
problems. For these reasons, the SA was chosen in the present study and applied to the study of the three-cable “BSL
network”, installed on the Fred Hartman Bridge (Houston, Texas, USA). Therefore, the objectives of this study are:

1) To find the mean value of the stochastic performance coefficient $\Delta K_1$ of the BSL network for each in-plane “mode” due to
uncertain vibration amplitude.
2) To compare results obtained via efficient SA algorithms to brute force Monte Carlo simulations (see e.g., [37]) needed to
generate a meaningful distribution of the vibration amplitude and to determine the exact statistical properties of $\Delta K_1$.

The paper is organized as follows. Section 2 summarizes the previous study [1] and includes descriptions of both the
deterministic and the stochastic model. Section 3 introduces the SA and the Monte-Carlo sampling method while Section 4
presents the numerical results. The conclusions are provided in Section 5. A supplementary study is given in Appendix A to
further justify some assumptions used by the numerical algorithm.

2. Description of the mathematical models and of the benchmark system

2.1. Deterministic model

In a previous study [1], the ELM was used to determine the influence of a unilateral restoring-force mechanism in the
cross-tie of a cable network and to evaluate the performance of the system in the presence of deterministic vibrations. The
main aspects of the model are briefly summarized in this section. Then, the formulation is extended to the case of random
vibration amplitude.

The model for stay vibration is based on the linear taut-cable theory by Irvine [23]; a short description of the main
assumptions is provided here for the sake of completeness. The model neglects localized flexural stiffness effects of a stay at
the anchorages with the deck or in the proximity of the connector with cross-tie and the effects of geometric nonlinearity (sag effect). The relevance of geometric nonlinearity can significantly affect the dynamics of “stand-alone” cables; in certain conditions vibrations of an individual stay can become unstable, leading to “whirling” motions (e.g., [38,39]). Nevertheless, it has been argued that, in a cable-cross-tie system, the large tension in the stays, combined with the pre-tensioning effect imparted to the cross-ties (transversely attached to the stays), is capable of regularizing the sag effects in the primary system [40]. The modest impact of the sag effect was also confirmed by an experimental and analytical study [26], which considered a system composed of two main parallel cables and a secondary cable. The results demonstrated that the main source of nonlinear dynamics for the whole system is due to the secondary cable (i.e., the cross-tie). The nonlinear behavior can cause either a dependence of vibration frequency on the amplitude or (in very special cases) an internal resonance [26]. Nevertheless, the influence of the sag in the main cables was clearly shown to be of minor importance.

Out-of-plane vibration is not considered, as explained in Section 1. The partial differential equations of motion of the taut-cables are simultaneously solved for the “frequency” and the “modes” (by ELM) by enforcing a series of compatibility and equilibrium conditions at the interfaces between each stay and each cross-tie segment. By setting linearized solutions, this framework leads to an algebraic system of equations, equivalent to an eigenvalue/eigenvector problem.

A three-cable system, installed on the Fred Hartman Bridge [24,27] and labeled as BSL network, is considered in this study as a benchmark case (Fig. 1). The system is composed of three stays (BS13, BS14 and BS15) of the “B-line” on the south tower of the bridge and one cross-tie [1]. The stay properties description can also be found in Table 1 of Ref. [1]. The reference stay is BS15 with index \( j = 1 \).

The source of nonlinearity is exclusively present in the constitutive law, describing the cross-tie restoring mechanism. Nonlinearity is used to simulate the unilateral restoring force effect in the cross-tie segment installed between \( P_{1,1} \) and \( P_{2,1} \) (connecting the stays BS15 with \( j = 1 \) and BS14 with \( j + 1 = 2 \)). Linear elastic behavior is postulated in the restrainer located between \( P_{2,1} \) and \( P_{3,1} \), connecting BS14 with BS13 in Fig. 1.

The restoring force in the connector segment was designated as \( F_{j+1} = F_{1,2} \) to simulate the interaction offered by this restrainer between \( P_{1,1} \) and \( P_{2,1} \), which connects stays \( j = 1 \) and \( j + 1 = 2 \). The cross-tie is positioned at \( x_{j+1} = \delta_{1,1} k_{j} \) on stay \( j = 1 \). This force is given by

\[
F_{j+1} = \begin{cases} 
  k_{j} (y_{j+1,1} - y_{j+1,1}) & \text{if } (y_{j+1,1} - y_{j+1,1}) > 0 \\
  k_{j} (y_{j+1,1} - y_{j+1,1}) & \text{if } (y_{j+1,1} - y_{j+1,1}) < 0 \land |F_{j+1}| < F_{0j} \\
  0 & \text{otherwise.} 
\end{cases}
\]

The linear behavior of the cross-tie segment, located between points \( P_{j,1} = P_{1,1} \) and \( P_{j+1,1} = P_{2,1} \), is measured through the dimensionless stiffness parameter \( d_{kj} = T_{j}/(k_{j} L_{j}) \) [1].

In the case of the BSL network, the following conventions are used in this section and in the remainder of this paper. The position of the restrainer is \( x_{1,1} = 0.52L_{1} \) (with \( L_{1} = 76.5 \text{ m} \)). In the linear case (no unilateral behavior), the same linear stiffness properties in each of the two segments of the cross-tie, with \( d_{k1} = d_{k2} = d_{k3} \) are given by \( d_{k1} = T_{1}/(k_{1} L_{1}) \). The level of pre-stressing in the connector [1]:

\[
\tau_{0j} = \frac{F_{0j}}{T_{j}},
\]

where \( F_{0j} \) is the initial pre-tensioning force in the cross-tie segment, installed between stay \( j = 1 \) (BS15) and \( j + 1 = 2 \) (BS14). The ratio \( F_{0j}/k_{j} \) designates the initial pre-tensioning relative deformation relative of the connector.

The application of ELM to the BSL network yields an equivalent linearized dimensional stiffness of the spring-type model in cross-tie segment between \( P_{1,1} \) and \( P_{2,1} \) (with \( k_{ELM} = k_{ELM,1} \) in generalized form). This quantity depends on the ELM

Fig. 1. Prototype three-cable network, installed on the F. Hartman Bridge [12] (schematic) – the cross-tie segment between stays BS15 and BS14 is nonlinear.
normalized frequency $\alpha E$ and the dimensionless vibration amplitude parameter $\lambda$ as follows [1]:

$$k_{ELM,j}(\alpha E, \lambda) = \frac{k_j}{2} \left[ 1 + \frac{r_0^2 d_{kj}}{\lambda^2 (\delta_j S_{j1}(\alpha E) - \delta_j + 1 S_{j1+1+1}(\alpha E))^2} \right],$$  \(3\)

where $\delta_j$ is the ratio among modal amplitudes for the specific mode and $S_{j1} = \sin(\alpha E f_j \delta_j)$. The quantity $S_{j1}$ is a trigonometric function that is derived from the set of compatibility and equilibrium equations enforced at the anchorage points of the cross-tie. As shown in our previous work [1], this quantity was found after imposing that, according to the ELM solution, the function that is derived from the set of compatibility and equilibrium equations enforced at the anchorage points of the cross-tie was introduced as

$$Y_{ELM,j}(\lambda) = A_{ELM,j} \sin(\alpha E f_j \lambda_j / k_j)$$

with frequency ratio $f_j$. The quantity $S_{j1}$ is used to measure the modal displacement relative to modal amplitude $A_{ELM,j}$ at the position of the cross tie. This information was later employed while equating the work done by the nonlinear cross-tie with unilateral effect to the work done by the equivalent linear cross-tie (ELM). The complete derivation may be found in Eq. (14) of Ref. [1].

Eq. (3) was subsequently modified to find an equivalent dimensionless stiffness parameter $d_{ELM,j} = T_j/(k_{ELM,j} E)$, based on $k_{ELM,j}$, which simulates the occurrence of the slackening. The inverse of the stiffness parameter is

$$\frac{1}{d_{k_{ELM,j}}(\alpha E, \lambda)} = \frac{1}{2d_{kj}} \left[ 1 + \frac{r_0^2 d_{kj}}{\lambda^2 (\delta_j S_{j1}(\alpha E) - \delta_j + 1 S_{j1+1+1}(\alpha E))^2} \right].$$  \(4\)

Eqs. (3) and (4) imply that the following relation is satisfied:

$$r_0^2 d_{kj}^2 / \lambda^2 \leq (\delta_j S_{j1}(\alpha E) - \delta_j + 1 S_{j1+1+1}(\alpha E))^2.$$

In order to assess the malfunctioning in the cross-tie segment, installed between stay $j = 1$ (BS15) and $j + 1 = 2$ (BS14), the “performance coefficient” of the cross-tie was introduced as

$$\Delta K_j(\alpha E, \lambda) = \frac{k_{ELM,j}(\alpha E, \lambda)}{k_j} = \frac{1}{2} \left[ 1 + \frac{r_0^2 d_{kj}}{\lambda^2 (\delta_j S_{j1}(\alpha E) - \delta_j + 1 S_{j1+1+1}(\alpha E))^2} \right].$$  \(5\)

The performance coefficient $\Delta K_j = \Delta K_1$ in Eq. (5) is an indicator of the malfunctioning of a cross-tie during vibration at the amplitude $\lambda$. The coefficient varies between 0.5 in case of complete malfunctioning of the connector ($r_0 = 0$) and 1.0 in the case of linear behavior. All the constitutive parameters defined in Eqs. (3)-(5) depend on the unknown frequency $\alpha E$ and the amplitude parameter $\lambda$. In Eq. (5) the dependence on $\lambda$ is implicit, since $\alpha E$ is also a function of $\lambda$ because of Eq. (4).

After applying the ELM, the solution can be determined by an equivalent eigen-value/eigen-vector problem in terms of $\alpha E$ and “mode shapes”. This problem can be cast in a compact formula $Q(\alpha E)\Psi = \Psi = 0$. The unknown eigen-value $\alpha E$ associated to the eigen-vector $\Psi$ becomes root of the characteristic equation $\det(Q) = 0$. Moreover, the internal energy transmitted through the connector increases with the performance coefficient as $\Delta K_j \rightarrow 1$.

### 2.2. Stochastic model

In the stochastic case, Eqs. (3)-(5) are affected by uncertainty (Section 1.2) through the parameter $\lambda$. Therefore, $\lambda$ becomes a stochastic variable, $Q$ becomes a random matrix and $\Psi$ is a random eigen-vector collecting the amplitude of the mode-shape functions. Thus, stochastic methods must be employed to find the roots $\alpha E$ of the characteristic polynomial $Q = \det(Q)$. Interestingly, a one-to-one relationship in Eq. (5) can be utilized to find the expected value and the variance of $\Delta K_j = \Delta K_1$ in terms of the roots $\alpha E$, mode by mode.

The main objective of this study is the estimation of the expected value of the performance coefficient $\Delta K_j = \Delta K_1$, averaged on the stochastic variable $\lambda$. Clearly, the dependence of $\Delta K_j$ on $\lambda$ in Eq. (5) is implicit, since the normalized frequency $\alpha E$, which is not initially known, is a function of $\lambda$. The averaging on $\lambda$ can be well captured by Eq. (5) and by using $\pi E$, which is the root of the characteristic polynomial $Q_0$, averaged over all the values of $\lambda$ distribution. The estimation of $\pi E$ was carried out by employing the SA based on the Robbins–Monro theorem [29–31]. The SA and its Monte-Carlo validation are described in Section 3.

### 3. Use of Monte-Carlo sampling and Stochastic Approximation to determine $\Delta K_j = \Delta K_1$

#### 3.1. Monte-Carlo sampling

The stochastic problem was initially solved by Monte Carlo sampling. The randomness of $\lambda$ can be represented by a random sequence of the amplitude parameter $\{\lambda_1, \lambda_2, ..., \lambda_q, ..., \lambda_m\}$, which generates a corresponding random series of “performance coefficients” $\{\Delta K_{j1}, \Delta K_{j2}, ..., \Delta K_{jq}, ..., \Delta K_{jm}\}$ for the connector segment installed between stays $j$ and $j + 1$. The latter vector reorders the random performance of the cross-tie segment, which was obtained by solving the problem for each $\lambda_q$ of the sample of dimension $q (q = 1, ..., m)$.

A uniform probability distribution between $\lambda = 0$ and $\lambda = \lambda_0$ (“upper limit”) was used to describe the random variable $\lambda$. This hypothesis reproduces the epistemic uncertainty and preserves simplicity in the analysis. The uniform probability
distribution best represents the current state of knowledge of the various vibration regimes and the fact that limited
information on amplitudes, experienced by a cable-cross-tie system at full scale during aeroelastic oscillation, is available.
Moreover, since the parameter $\lambda$ is not an intrinsic noise, the use of other probability distributions such as the Gaussian
distribution, might be not a good choice.

When the brute force Monte Carlo sampling is employed, a sequence of $\Delta K_{j,q}$ ($j=1$) can be found numerically from the
sequence of $\lambda_q$ by ELM. The expected value $\Delta K_j$ can be estimated from the $\Delta K_{j,q}$ sample as

$$
\Delta K_j = \lim_{m \to \infty} \frac{1}{m} \sum_{q=1}^{m} \Delta K_{j,q}
$$

Various sample sizes $m$ were used in order to approximately estimate the limit in Eq. (6) for sufficiently large $m$. The
mean of the distribution, $\Delta K_j = \Delta K_1$, can be employed as an indicator of efficiency of the connector. The larger the deviation
of $\Delta K_1$ from the upper limit 1.0, the larger is the imbalance of the relative distribution.

As an example, Fig. 2b shows the histogram of the “performance coefficient” $\Delta K_1$ ($j=1$) for the third-mode of the BSL
network with $d_k=0.018$ and $\tau_{0,1}=0.015$ and the uniformly-distributed random input $0 \leq \lambda \leq \lambda_u$ (Fig. 2a). In Fig. 2, the sample
size is 5000. The system parameters were described in Section 2.1. The upper limit $\lambda_u=0.005$ was used. This value
corresponds to an amplitude equal to 1/200th of the length of BS15 with $L_1=76.5$ m and it also corresponds to a vibration of
about one or two cable diameters, compatible with large-amplitude aeroelastic vibration (e.g., [3]).

The histogram in Fig. 2b suggests that the distribution of $\Delta K_1$ is predominantly bi-modal, with the large peak around
$\Delta K_1=1$ being generated to a negligible influence of very small vibration (lower values of the random $\lambda$ sequence) on the
malfunctioning of the cross-tie. This subset of values leads to a “perfectly linear” behavior with $\Delta K_1=1$.

Fig. 3 shows an example of two recurrence histograms of the performance coefficient for the third mode of the BSL
network, found by modifying the level of pre-stressing. The histogram of $\Delta K_1$ is bounded since the random variable is
defined between 0.5 and 1.0. Fig. 3a illustrates the case of a low pre-stressing force ($\tau_{0,1}=0.005$), where the bi-modal
distribution of $\Delta K_1$ is unbalanced toward the left-side limit of the graph with a mean value approximately equal to

![Fig. 2. Histogram of random amplitude parameter $\lambda$ ($0 \leq \lambda \leq \lambda_u=1/200$) and corresponding distribution of performance coefficient $\Delta K_1$ for the BSL network on the Hartman Bridge for $d_k=0.0018$ and moderate level of pre-stressing ($\tau_{0,1}=0.0025$). Results correspond to the second “equivalent mode” of the system.](image-url)
A different behavior in the cross-tie can be noted in the case of larger initial pre-stressing, illustrated in Fig. 3b, where the mean of the distribution $\Delta K_1$ is shifted toward the right side of the histogram. In Fig. 3, the sample size is 5000.

### 3.2. Stochastic Approximation

The SA algorithm [5,7] was employed to construct a computationally efficient estimator of $\Delta K_j = \Delta K_1$, obtained by re-writing Eq. (5) as

$$\overline{\Delta K_j} \approx \Delta K_j(\pi_E) = \frac{k_{\text{MOD},j}(\pi_E)}{k_j} \quad (7)$$

From the inspection of Eq. (7) it can be noted that the stochastic problem for $\Delta K_j$ can be approximately re-formulated by first seeking for the average of $\pi_E$, the amplitude-dependent random frequency of the “equivalent modes” of the cable network. The corresponding value of $\overline{\Delta K_j} = \overline{\Delta K_1}$ can be found by evaluating Eq. (7) in correspondence with this value.

In the algorithm, the random sequence $\{\lambda_1, \ldots, \lambda_q, \ldots, \lambda_m\}$, defined above, generates a corresponding sequence of equivalent frequencies $\{\alpha_{E,1}, \ldots, \alpha_{E,q}, \ldots, \alpha_{E,m}\}$, obtained by solving the equivalent eigen-value/eigen-vector problem. The roots of the polynomial $Q_\lambda(\alpha_E)$ are indicated as $Q_\lambda(\alpha_{E,q}) = 0$. As in Eq. (6), the average value of the ELM frequency $\pi_E$ can be estimated from the $\alpha_{E,q}$ sequence as

$$\pi_E = \lim_{m \to \infty} \frac{1}{m} \sum_{q=1}^{m} \alpha_{E,q}. \quad (8)$$

The SA method [31] can be employed to find $\pi_E$. In general, the SA finds the root of a function $g$ of a generic independent variable $\alpha$, affected by uncertainty simulated by a white noise “error” $e_q$. Thus, an observation of $g$ is given by

$$Y_q(\alpha) = g(\alpha) + e_q(\alpha). \quad (9)$$

Fig. 3. Histogram of $\Delta K_1$ for the third “equivalent mode” of the BSL network with $d_k = 0.018$ due to random amplitude parameter $\lambda$ with uniform distribution ($0 \leq \lambda \leq \lambda_u = 1/200$) and various levels of pre-tensioning: (a) $\tau_{0,1} = 0.005$ (low level of pre-tensioning), (b) $\tau_{0,1} = 0.025$ (high level of pre-tensioning) – a different vertical-axis scale was used to highlight variation in the distributions.
According to the Robbins–Monro theorem a root of $g$ can be found using the recursive formula given by [29,31]

$$a_{q+1} = a_q - a_q Q_q(a_q),$$

(10)

where the damping term $a_q$ must satisfy the Robbins–Monro conditions [30]

$$\sum_{q=1}^{\infty} a_q^2 < \infty \text{ and } \sum_{q=1}^{\infty} a_q = \infty.$$

(11)

A possible choice for $a_q$ can be written as [29]

$$a_q = \frac{a}{(q+1)^{\delta_{SA}}}$$

(12)

where $0.5 < \delta_{SA} < 2.0$ and $a$ are arbitrary constants that can be chosen to accelerate the convergence.

The simplest application of the SA algorithm is the recursive calculation of the mean $\bar{\pi}_E = \pi_{E_{\infty}}$, namely

$$\bar{\pi}_{E,q+1} = \bar{\pi}_{E,q} - \frac{1}{q+1}(\bar{\pi}_{E,q} - \bar{\pi}_{E,q+1}).$$

(13)

In this case $a=1$, $\delta_{SA}=1$. Moreover the noise-less function is $g(\alpha) = \alpha - \bar{\pi}_E$ i.e., the deviation (distance) of $\alpha$ from its true value $\pi_\alpha$ while $Q_q(\bar{\pi}_E) = \bar{\pi}_{E,q} - \bar{\pi}_{E,q+1}$ is the noisy observation of $g$.

In order to find $\bar{\pi}_E$ and to avoid expensive root finding for many characteristic polynomials needed at each step of Eq. (13), an estimator provided by the root of the average characteristic polynomial is considered:

$$\bar{Q}(\alpha) = \lim_{m \to \infty} \frac{1}{m} \sum_{q=1}^{m} Q_{\lambda_q}(\alpha).$$

(14)

The root of $\bar{Q}(\alpha)$ is determined with the SA from Eq. (10), in which the assignment $Q_{\lambda_q} = \bar{Q}_{\lambda_q}$ is used to designate an imperfect (uncertain) observation of the original function $g = Q(\alpha)$, i.e., evaluated at step $q$ by truncating the limit of sequence in Eq. (14) after the first $q$ steps. Therefore the recursive equation becomes

$$\bar{\pi}_{E,q+1} = \bar{\pi}_{E,q} - a_q Q_{\lambda_q}(\bar{\pi}_{E,q}).$$

(15)

where the sequence $\lambda_q$ is generated by a random number generator for the uniform distribution. In Eq. (15) partial estimation of a root at step $q$ is used in the next step. Therefore we avoid to solve the eigenvalue problem for each $\lambda_q$ of the random sequence. The root of $\bar{Q}(\alpha)$ is a suitable estimator for the true mean of the frequency if these two quantities coincide within a few percent. For problems analyzed herein, the values $a=0.25$ and $\delta_{SA}=0.95$ yield good convergence properties of the algorithm.

Fig. 4 shows an example of the convergence of the SA for the case illustrated in Fig. 3a. In Fig. 4a the histogram, which illustrates the distribution of the random equivalent frequency $\alpha_E$, is evaluated by brute force Monte Carlo sampling using 500 realizations. The average value found by Monte-Carlo sampling is also shown in the figure.

Fig. 4b illustrates the calculation of the average frequency $\bar{\pi}_{E,q}$ (recursive-sample-average-curve in the figure), when the sequence $\lambda_q$ is chosen according to the uniform random distribution within the limits $0 \leq \lambda \leq \lambda_a = 1/200$. The first curve in the figure (thin continuous line), illustrates the estimation of the average $\bar{\pi}_{E,q}$ by brute force sampling as a function of the iteration number $q$. A new root $\alpha_{E,q+1}$ can be numerically found at each step, using the determinant of the characteristic polynomial and the element $\lambda_{q+1}$ of the sequence. The root $\alpha_{E,q+1}$ is found from the condition $Q_{\lambda_{q+1}}(\alpha_{E,q+1}) = 0$. The average $\bar{\pi}_{E,q+1}$ is subsequently updated according to Eq. (13). The ideal (true) average is the asymptote of this curve as $q$ tends to infinity.

The results of the SA search algorithm are also shown in Fig. 4b (thick solid line). The SA algorithm starts from a pre-assigned initial value $\alpha_0$, needed by the recursion formula, which can be set as the corresponding linear root of the polynomial for the given mode. In Fig. 4b the difference between the two curves is very small after 200 iterations. Moreover, the relative error between the estimator, evaluated by Eq. (15) at iteration $q$, and the “true average” $\bar{\pi}_E$, which is numerically assessed by Monte Carlo methods by collecting a larger sample of 500 roots from $Q_{\lambda_{q+1}}(\alpha_{E,q}) = 0$, is less than 1 percent. The difference between the two curves becomes approximately constant after 200 iterations in Fig. 4b. It is however important to carefully select the coefficients in Eq. (12) to avoid convergence problems by conducting a series of preliminary numerical tests.

In the next section, we shall confirm that Eqs. (7) and (15) also yield an excellent estimation of $\Delta \Sigma K_j$ (and $\Delta \Sigma K_1$) with the SA method for most of the relevant parameter space.

It must also be noted that the use of Eq. (15) to estimate the mean value of the frequency and, consequently, $\Delta \Sigma K_j$ (and $\Delta \Sigma K_1$) is based on the hypothesis defined by Eq. (14). This fact implies that the result is acceptable if the following relationship between the expected values, used in Eq. (7) approximately holds: $\Delta \Sigma K_j = E[\Delta \Sigma K_j(\alpha_E)] \approx \Delta \Sigma K_j(E[\alpha_E]) = \Delta \Sigma K_j(\bar{\pi}_E)$. Clearly, this approximation crucially depends on the nonlinearity of the function $\Delta \Sigma K_j$ in Eqs. (5) and (7). This further approximation error, introduced in Eq. (7), will be later investigated in Appendix A.
4. Numerical results

4.1. Introduction

The SA algorithm was deployed to study the BSL network under stochastic disturbance, simulated through a random variable \( \lambda \) uniformly distributed between 0 and \( \lambda_u = 0.005 \). The upper limit \( \lambda_u \) corresponds to an amplitude of 1/200th of the reference cable length.

The presence of vibrations with random amplitude \( \lambda \) induces a distribution for the performance parameter \( \Delta K_1 \). The performance is still adequate [1] for small-amplitude vibration and high pre-tensioning forces. Imparting a high level of pre-stressing to the restrainer is not needed as it could also indirectly produce undesirable sag on the stays (e.g., [40]). Therefore, it is important to quantify, for a random vibration regime \( \lambda \), the minimum level of pre-stressing \( \tau_{01} \) that maximizes \( \Delta K_1 \) and ensures a linear behavior in the BSL cable network. The mean value \( \Delta K_1 \) of the random \( \Delta K_1 \) was investigated to characterize the acceptable \( \tau_{01} \) values for the BSL network. Since the evaluation of \( \Delta K_1 \) requires a large number of Monte-Carlo simulations (e.g., from Fig. 3), the efficient numerical technique, based on the SA, was utilized as an alternative to find \( \Delta K_1 \).

The parameters of the SA algorithm along with initial search point of the recursive sequence were calibrated in order to obtain the desired accuracy (e.g., Fig. 4b) and a smooth convergence.

4.2. Verification of the SA algorithm by comparison against Monte Carlo sampling

Figs. 5 and 7 illustrate the comparison of the results obtained by both SA method and Monte Carlo sampling. Two distinct values of stiffness parameter \( d_k \) are chosen to represent two limiting cases. The first case corresponds to a very rigid connector while the second one to a moderately flexible connector. The stiffness, tension force and cable length values, used to define \( d_k \), are compatible with realistic stay configurations on the actual bridge [1].

In Figs. 5 and 7, the results are presented for the second (Mode II) and third (Mode III) equivalent modes of the BSL network as a function of the pre-tensioning parameter \( \tau_{01} \). In each figure the linear solution, which neglects the unilateral behavior in the cross tie, is shown as a reference (\( \Delta K_1 = 100 \) percent). Also, the “exact” curves, obtained by brute force Monte Carlo sampling with 5000 realizations, are illustrated by thick continuous lines with a filled circle marker and are designated by the label “B.F.” in the legend, with the number in parenthesis indicating the sample size. These curves were compared to those estimated by SA method, which...
Fig. 5. Expected value of performance coefficient for the second mode of the BSL network with uniformly-distributed random amplitude $0 \leq \lambda \leq (1/200)$: (a) flexible connector ($d_k=0.018$) and (b) rigid connector ($d_k=0.0018$) – "B.F.", Monte Carlo sampling; "R.M.", SA method.

Fig. 6. Relative error in the estimation of $\Delta K_1$ by SA method for the second mode of the BSL network in Fig. 5: (a) flexible connector ($d_k=0.018$) and (b) rigid connector ($d_k=0.0018$) – the "B.F." results are used as "exact" values.
are depicted as thick continuous lines with an empty circle marker and are labeled as “R.M.” in the legend. These curves can be interpreted as the average performance curves, conditional on the value of $\tau_{0,1}$ and for a random amplitude $0 \leq \lambda \leq 1/200$. In order to analyze the influence of a random $\lambda$, the “B.F.” and “R.M.” curves were examined in relation to the deterministic solutions exhibiting unilateral behavior with constant amplitude compatible with the hypothesis $0 \leq \lambda \leq 1/200$. The deterministic solution was obtained [1] by selecting three intermediate values in the interval $0 \leq \lambda \leq 1/200$, $\lambda = 1/1000$, $\lambda = 1/400$ and $\lambda = 1/200$. These three curves are depicted in the figures by dashed lines of various thicknesses and accompanied by labels “(ELM)” in the legend. As discussed in Ref. [1], it is of particular interest from the design standpoint the identification of a minimum level of pre-tensioning $\tau_{0,1}$, which avoids the slackening in the restrainer.

Fig. 5 shows the $\Delta K_1$ results for the second mode of the BSL network as function of $\tau_{0,1}$ in the case of a flexible connector with $d_k = 0.018$ (Fig. 5a) and in the case of a rigid connector with $d_k = 0.0018$ (Fig. 5b). Results obtained by SA algorithm are in a good agreement with those using Monte Carlo sampling. As shown in Fig. 6, the relative error in absolute terms is smaller than 7 percent in both cases.

Fig. 7 illustrates the behavior of the third mode of the BSL network. The agreement of the results obtained by Monte Carlo and SA methods is very good for flexible cross-tie (Fig. 7a), as observable from the analysis of the estimation error in Fig. 8a. In fact, the relative error is consistently less than 4 percent. Discrepancies between the two methods only emerge in the case of a rigid connector in Fig. 7b, since the SA method (“R.M.” continuous line with empty circle marker) tends to suggest a larger performance for $\tau_{0,1} > 0.2$ compared to the values obtained by Monte Carlo sampling. The maximum estimation error, as shown in Fig. 8b, is, however, still of the order of 6 percent.

These results indicate that the SA algorithm can be successfully employed to estimate the expected value of the performance coefficient, $\Delta K_1$, as a function of pre-tensioning parameter and in the presence of a random vibration amplitude $\lambda$. If the average value of the random $\Delta K_1$ was employed as a performance indicator, the probabilistic setting with random $\lambda$ consistently shows a better performance than the corresponding deterministic case with amplitude equal to the upper limit $\lambda_u$ of the uniform distribution in the range of $\tau_{0,1}$, where the behavior rapidly deteriorates. This deterioration depends on $d_k$ and the mode that is considered: for example $0.01 < \tau_{0,1} < 0.03$ in Fig. 7a for the third mode. This probabilistic effect, can be on occasion remarkable, 20 percent or larger at about $\tau_{0,1} = 0.02$ in most cases exhibited by Figs. 5 and 7.

4.3. Parametric study to investigate the influence of $d_k$ and $\tau_{0,1}$ on $\Delta K_1$

A parametric study was carried out to understand the stochastic-vibration behavior of the BSL network by taking into account the nonlinearity in the cross-tie through the combined effect of $d_k$ (dimensionless linear stiffness parameter) and $\tau_{0,1}$ (pre-tensioning parameter). This investigation was conducted by using the SA algorithm to estimate $\Delta K_1$. 

![Graph](image-url)

**Fig. 7.** Expected value of performance coefficient for the third mode of the BSL network with uniformly-distributed random amplitude $0 \leq \lambda \leq (1/200)$: (a) flexible connector ($d_k = 0.018$) and (b) rigid connector ($d_k = 0.0018$). - “B.F.”, Monte Carlo sampling; “R.M.”, SA method.
Fig. 9 summarizes the result of this analysis and shows various trends of the performance coefficient for the BSL-network fundamental modes. In this simulation, the pre-tensioning parameter \( \tau_{0,1} \) was taken as variable between 0 and 0.05 and four distinct values of the stiffness parameter \( d_k \) were examined, between \( d_k = 0.018 \) and \( d_k = 0.0018 \), coincident with the two limiting conditions for the cross-tie (rigid and flexible). For each \( d_k \), a \( \tau_{0,1} \) vs. \( \Delta \kappa_1 \) curve was determined.

Fig. 9a illustrates the behavior of \( \Delta \kappa_1 \) for the second mode of the BSL network. If the stiffness parameter \( d_k \) was selected a priori (by design), it would possible to indicate a minimum level of \( \tau_{0,1} \) to ensure adequate dynamic performance. Fig. 9a also suggests that, in the case of second mode, a lower pre-tensioning force in the restrainer is required for satisfactory performance (e.g., \( \Delta \kappa_1 \) above 90 percent) if a cross-tie with larger stiffness (lower \( d_k \)) was utilized. The pre-tensioning force needed to reach a fully linear behavior of the cross tie (\( \Delta \kappa_1 = 100 \) percent) is less than half, in case of a rigid connector. As shown in Fig. 9a the beneficial effects of the pre-tensioning force are more evident for a stiffer connector compared to a flexible one; in other words, a small increment of restrainer stiffness (from \( d_k = 0.0018 \) to \( d_k = 0.0020 \)) would be sufficient to modify the “nonlinearity limits” (minimum \( \tau_{0,1} \) associated with \( \Delta \kappa_1 = 100 \) percent; also refer to Section 5.2 in [1]) from 0.015 to 0.025.

A different behavior is observed in Fig. 9b for the third mode of the BSL network, where, for a more rigid connector (\( d_k = 0.0018 \)), it would necessary to increase the pre-tensioning parameter to 0.03 in order to achieve a fully linear behavior in the cross-tie. In the case of a more flexible connector (\( d_k = 0.0018 \)), the pre-tensioning parameter necessary to ensure \( \Delta \kappa_1 = 100 \) percent must be greater than 0.04. In contrast with the second mode, the stiffness parameter \( d_k \) has a limited influence on the performance of the connector and the network. It must be noted that this effect is on occasion overestimated by the SA method, which “feels” a larger increment in the performance coefficient \( \Delta \kappa_1 \) not always compatible with Monte-Carlo sampling results, as shown in Fig. 6b by comparing the SA results (continuous line with empty circle marker) with the BF curve (continuous line with filled circle marker) for \( \tau_{0,1} > 0.2 \).

4.4. Estimation of the variance of \( \Delta \kappa_1 \) by Monte Carlo sampling

In order to complete the probabilistic analysis for the BSL network, the variance of the equivalent frequency \( \alpha_e \) in the presence of unilateral behavior of the cross-tie was computed by Monte Carlo methods. Both equivalent Mode II (Fig. 10) and Mode III (Fig. 11) were investigated along with the effect of a stiffness variation in the connector, from flexible with \( d_k = 0.018 \) to rigid with \( d_k = 0.0018 \). A limited influence of sample size on the variance estimation was observed, if a sample with 500 realizations (dashed lines in Figs. 10 and 11) was utilized in comparison with a more accurate assessment using 5000 sample points. This verification was carried out at selected values of \( \tau_{0,1} \) only, i.e., at the points denoted by a circle marker in the same figures. As a consequence, the dashed lines in Figs. 10 and 11 were considered as “exact” results for the variance of \( \Delta \kappa_1 \) and employed in the analysis of the relationship between \( \tau_{0,1} \) and the variance of \( \Delta \kappa_1 \).
Fig. 9. Influence of a variation of $d_k$ on the expected value of the performance coefficient for the BSL network with pre-stressing parameter $0 \leq r_{0.1} \leq 0.05$ for uniformly-distributed random vibration amplitude with $0 \leq \lambda \leq (1/200)$.

Fig. 10. Variance of the random performance coefficient for the second mode of the BSL network with uniformly-distributed random amplitude $0 \leq \lambda \leq (1/200)$: (a) flexible connector ($d_k=0.018$) and (b) rigid connector ($d_k=0.0018$) – results are obtained by Monte-Carlo sampling only ("B.F.").
The comparison between Figs. 10 and 11 suggests that the order of magnitude of the variance is similar in both Modes II and III. The peak value of the variance tends to slightly decrease in the case of the second mode with a rigid connector ($d_k = 0.0018$ in Fig. 10b). Even in this case, the variance analysis tends to confirm the results shown in Figs. 5 and 7. The vanishing of the variance can be in fact interpreted as a corresponding performance coefficient equal to 100 percent, for deterministic vibration with $\lambda = 0.005$.

5. Concluding remarks

The SA was applied to the solution of generalized amplitude-dependent in-plane vibration “modes” of cable networks, affected by stochastic aeroleastic vibrations. Nonlinearity in the cross-ties was used in conjunction with a random amplitude parameter, which was employed to describe the uncertainty in the estimation of aeroleastic vibration mechanisms related to wind-induced and wind-rain-induced excitation in the stays. The SA method has allowed an efficient estimation of the average dynamic performance of a benchmark three-cable network in a stochastic setting. The definition of dynamic performance was based on the performance coefficient of the cross-tie, originally introduced in a previous study. The verification of the SA estimator was carried out by Monte Carlo sampling. Moreover, a preliminary estimation of the variance of the performance coefficient was also discussed.

The comprehensive parametric study presented in this investigation provides crucial evidence of the combined effects of nonlinear cross-tie behavior and stochastic dynamics. In particular, numerical results suggest that the same level of nonlinearity (i.e., vibration amplitude) can induce non-negligible variations on the mean value of the performance coefficient, especially for the second mode of the modeled system, depending on the tension in the cables and the pre-tensioning force in the cross-tie. The main probabilistic effect is the fact that a considerably lower pre-tensioning force can be applied internally to the cross-tie to ensure the same average restraining effect needed to prevent slackening. Clearly, small values of $\lambda$, belonging to the uniform-distribution, significantly contribute to the adequate performance of the cable network.

It must be noted that the SA method involves some approximation, which can be alleviated by limiting the variability in the amplitude parameter and by controlling the approximation error in the estimation of the mean value of the “equivalent” modal frequencies (see the Appendix for a detailed discussion). Nevertheless, the advantage of the SA is that it exclusively requires evaluation of an initial “seed” frequency, and avoids finding the roots of many characteristic polynomials since the average frequency solution is obtained by subsequent iterations. Therefore, the SA considerably reduces the computational time. This merit is perhaps not noticeable in the present case, used as a proof of concept only, but it will become relevant for large cable networks composed of several stays and cross-ties, such as the system studied in [24]. Moreover, the SA algorithm, applied to estimate the mean value of a random quantity in this study, can also be used to evaluate the quantiles...
of the cumulative distribution of a random variable [29,30]. Thus we are opening new avenues for the computation of additional statistical descriptors, other than the mean value of the frequency, for studying unilateral free vibrations of cable networks. These aspects will be considered in future studies.

Acknowledgements

This study was supported by Northeastern University (NEU), Office of the Provost, “Tier-1 Seed Grant” for Interdisciplinary Research Projects in 2011–2014. The first author would also like to acknowledge the financial support of the “Regione Autonoma della Sardegna” (LR7 2010, Grant “M4” CRP-27585).

Numerical computations were performed on the cyber-grid of the NEU’s College of Engineering; collaboration with Techila Technologies, Ltd. of Finland is acknowledged.

Appendix A

As outlined in Section 3.2, the validity of the numerical results obtained by SA algorithm relies on the adequacy of Eq. (7) i.e., on the approximation $E[\Delta K_j(\alpha)] \approx \Delta K_j(E[\alpha])$. This appendix provides an analysis of this error. This approximation is investigated in the context of our benchmark, namely, the BSL network in Fig. 1. Since the nonlinearity is simulated in the cross-tie segment located between stay BS15 and BS14, the previous formula was analyzed with respect to the condition $j = 1$, $E[\Delta K_j(\alpha)] \approx \Delta K_j(E[\alpha]) = \Delta K_j(\bar{\alpha})$. Moreover, the input random variable $\lambda$ was assumed to be uniformly distributed in the interval $0 \leq \lambda \leq (1/200)$. This is the worst-case scenario, which takes into consideration previously published results [1]. Both second and third modes of the BSL network were examined. The error analysis requires the study of $\Delta K_j(\alpha)$ in Eq. (5). This function has a periodic component, which depends on the frequency $\alpha$, on the parameter $\lambda$ and, indirectly, on the pre-tensioning parameter $r_{0.1}$ (Eq. (2)). Even though this function is highly nonlinear, our parametric study suggests that, if one mode at a time is investigated, the randomness of $\lambda$ induces a relatively narrow distribution of the frequency $\alpha$. For example, we should bear in mind that $2.00 < \alpha < 2.03$ for Mode III in Fig. 4a, and that $1.89 < \alpha < 1.93$ for Mode II (figure not shown). Therefore, we can justifi the study of the function $\Delta K_j(\alpha)$ for $\alpha$ in the proximity of the mean value $E[\alpha]=\bar{\alpha}$, with $\bar{\alpha}=1.919$ for Mode II and $\bar{\alpha}=2.028$ for Mode III. In these neighborhoods the function $\Delta K_j(\alpha)$ can be expanded in Taylor series, truncated to the second order, such as

$$\Delta K_1(\alpha) \approx \Delta K_1(\bar{\alpha}) + \Delta K_1''(\bar{\alpha})(\alpha - \bar{\alpha}) + 0.5 \Delta K_1''''(\bar{\alpha})(\alpha - \bar{\alpha})^2$$  \hspace{1cm} (A.1)

with the first and second derivatives $d(\Delta K_j)/d\alpha|_{\alpha=\bar{\alpha}} = \Delta K_1''(\bar{\alpha})$ and $d^2(\Delta K_j)/d\alpha^2|_{\alpha=\bar{\alpha}} = \Delta K_1''''(\bar{\alpha})$ evaluated at $\alpha=\bar{\alpha}$. If $\alpha$ is random in the intervals above, taking the expectation on both sides of Eq. (A.1) leads to the following relationship, after few simplifications:

$$E[\Delta K_j(\alpha)] \approx \Delta K_j(\bar{\alpha}) + 0.5 \Delta K_1''''(\bar{\alpha})E[(\alpha - \bar{\alpha})^2] = \Delta K_j(\bar{\alpha}) + 0.5 \Delta K_1''''(\bar{\alpha})\text{var}(\alpha).$$  \hspace{1cm} (A.2)

with $\text{var}(\alpha) = E[(\alpha - \bar{\alpha})^2]$ the variance of the frequency. Eq. (A.2) enables the computation of the error $\epsilon_{\Delta K_1}$, induced by the approximation $E[\Delta K_j(\alpha)] \approx \Delta K_j(E[\alpha])]$: this is given by

$$\epsilon_{\Delta K_1} = \left|E[\Delta K_1(\alpha)] - \Delta K_j(\bar{\alpha})\right| \approx 0.5|\Delta K_1''''(\bar{\alpha})|\text{var}(\alpha).$$  \hspace{1cm} (A.3)

Similarly, the relative approximation error can be found from Eq. (A.3) and Eq. (A.1) and it is given as

$$\epsilon_{\Delta K_1,rel} = \frac{\epsilon_{\Delta K_1}}{E[\Delta K_1(\alpha)]} \approx 0.5\frac{|\Delta K_1''''(\bar{\alpha})|}{\Delta K_1(\bar{\alpha})}\text{var}(\alpha).$$  \hspace{1cm} (A.4)

Both $\epsilon_{\Delta K_1}$ and $\epsilon_{\Delta K_1,rel}$ are proportional to the local curvature of the function $\Delta K_1(\alpha)$ and the variance of the frequency. These quantities can be readily determined once $\text{var}(\alpha)$ is calculated mode by mode, e.g., by Monte-Carlo simulation as in Section 4.4 and Figs. 10 and 11. Inspection of Figs. 10 and 11 reveals that $\text{var}(\alpha)$ is proportional to the pre-tensioning parameter $r_{0.1}$ when $\lambda$ is uniformly distributed in the interval $0 \leq \lambda \leq (1/200)$. Moreover, the case of the BSL network with flexible connector ($d_k=0.018$) is clearly more delicate since larger values of $\text{var}(\alpha)$ can be observed by comparing the results at various $r_{0.1}$ for both modes. Therefore, we shall focus the error analysis on the configuration with $d_k=0.018$ only.

Prior to the study of $\epsilon_{\Delta K_1}$ and $\epsilon_{\Delta K_1,rel}$, the validity of Eq. (A.1) was examined in other ways. Fig. A.1a and b respectively illustrate the behavior of the function $\Delta K_1(\alpha)$ in the proximity of $\bar{\alpha}$ for Modes II and III of the BSL network with flexible connector ($d_k=0.018$). Both figures strongly confirm that in the two domains of $\alpha$ variation (important for both modes) the second-order Taylor series fit the function very well despite minor underestimations. Besides, the figures suggest that $\Delta K_1(\alpha)$ is regular and monotonic in these two domains. As a result, Eqs. (A.3) and (A.4) can be justified for the error evaluation.

Finally, Fig. A.2 illustrates the error analysis for Modes II and III for the BSL network with flexible connector ($d_k=0.018$), under the hypotheses listed above. The relative error $\epsilon_{\Delta K_1,rel}$ in Eq. (A.4) is shown in the figures, as a function of the pre-tensioning parameter $r_{0.1}$ (the corresponding $\text{var}(\alpha)$ was derived from Figs. 10 and 11). The error is almost zero for
Fig. A1. Performance coefficient $\Delta K_1$ as a function of dimensionless equivalent frequency $\alpha_E$ for the BSL network with uniformly-distributed random amplitude $0 \leq \lambda \leq (1/200)$ and flexible connector with $d_k = 0.018$: exact function from Eq. (5), first-order and second-order Taylor expansions about the mean frequency $\tau_{m}$ for: (a) Mode II with $\tau_{m} = 1.919$ and (b) Mode III with $\tau_{m} = 2.028$ (mean values are indicated by a circle marker).

Fig. A2. Relative approximation error ($\varepsilon_{\Delta K_1, rel}$, Eq. (A.4)) for the BSL network with uniformly-distributed random amplitude $0 \leq \lambda \leq (1/200)$ and flexible connector with $d_k = 0.018$: (a) Mode II and (b) Mode III.
Mode III in Fig. A.2b, whereas in the case of Mode II, the value of $e_{SA,rel}$ is about 2 percent. This figure confirms that the error corresponding to the approximation $E[\Delta K(E_{rel})] \approx \Delta K(E_{rel})$ is negligible and that the hypothesis used in Eq. (7) is clearly correct. As expected, the interval of $r_{0,1}$, at which $e_{SA,rel}$ is maximum in Fig. A.2a and b, coincides with the region of largest discrepancy between SA and Monte-Carlo method results. In particular, $r_{0,1}=0.01$ in Fig. A.2a coincides with the position of the largest peak in Fig. 6a, and the case $r_{0,1}=0.005$ in Fig. A.2b with the position of the largest peak in Fig. 8a.

References